FOURIER TRANSFORM AND INVERSE FOURIER TRANSFORM

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This short document will provide a few different forms of the Fourier transform and their inverse transforms, including a brief proof of how to get them.

1. Dirac delta function. Let $f : \mathbb{R} \to \mathbb{C}$. Let $\delta(\cdot)$ represent the Dirac delta function. The integral of the time-delayed Dirac delta function is

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)dt = f(T). \tag{1.1}$$

In particular, the Dirac delta function is an even distribution, in the sense that

$$\delta(-x) = \delta(x). \tag{1.2}$$

The Cauchy equation can be represented as

$$f(x) = \int_{-\infty}^{\infty} \delta(x-a)f(a)da, \tag{1.3}$$

where the delta function is represented as

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jp(x-a)}dp, \tag{1.4}$$

where j is the imaginary unit. Equation (1.4) is also equivalent to

$$\delta(x-a) = \int_{-\infty}^{\infty} e^{j2\pi p(x-a)}dp. \tag{1.5}$$

2. Fourier transform and its inverse transform. Let $\hat{f}$ be the Fourier transform of function f. Then one common convention defining the Fourier transform and its inverse is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi\xi x}dx, \tag{2.1}$$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{j2\pi\xi x}d\xi. \tag{2.2}$$

A brief proof from the Fourier transform (2.1) to (2.2) is given below. The proof from the inverse Fourier transform (2.2) to (2.1) is analogous.

Proof. Equation (2.1) is equivalent to

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(\xi)e^{j2\pi\xi y}d\xi &= \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x)e^{-j2\pi\xi x}dx)e^{j2\pi\xi y}d\xi && (2.3)\\
&= \int_{-\infty}^{\infty} f(x)(\int_{-\infty}^{\infty} e^{j2\pi\xi(y-x)}d\xi)dx && (2.4)\\
&= \int_{-\infty}^{\infty} f(x)\delta(y-x)dx \quad \text{.... (using (1.5))} && (2.5)\\
&= f(y) \quad \text{.... (using (1.2) and (1.3))}, && (2.6)
\end{aligned}$$

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which is (2.2) after replacing the variable y by x. $\square$

Another form defining the Fourier transform and its inverse is

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(x)e^{-jwx}dx, \tag{2.7}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{jwx}dw. \tag{2.8}$$

A brief proof from the Fourier transform (2.7) to (2.8) is given below.

Proof. Equation (2.7) is equivalent to

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(w)e^{jwy}dw &= \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x)e^{-jwx}dx)e^{jwy}dw & (2.9)\\
&= \int_{-\infty}^{\infty} f(x)(\int_{-\infty}^{\infty} e^{jw(y-x)}dw)dx & (2.10)\\
&= 2\pi\int_{-\infty}^{\infty} f(x)\delta(y-x)dx \quad \text{.... (using (1.4))} & (2.11)\\
&= 2\pi f(y) \quad \text{.... (using (1.2) and (1.3))}, & (2.12)
\end{aligned}$$

which is (2.8) after replacing the variable y by x and timing $1/2\pi$ from both sides. $\square$

Using the above proof, we can easily get other forms of the Fourier transform and the inverse, e.g.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-jwx}dx, \tag{2.13}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{jwx}dw. \tag{2.14}$$

3. How to remember the transforms. We can regard $\{e^{-j2\pi\xi x}\}$ or $\{e^{-jwx}\}$ with respect to ξ or $w \in (-\infty, \infty)$ as basis functions in the frequency domain, and with respect to x in the image/time space. Then the standard theorem in linear algebra tells us that any function f (or its transform $\hat{f}$) can be represented by these basis functions.

A more detailed reading can be found from
- Wikipedia `https://en.wikipedia.org/wiki/Fourier_transform`,
- WolframMathWorld `https://mathworld.wolfram.com/FourierTransform.html`,
- and the references therein.